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THE FUNCTON SPACE TOPOLOGY AS PRODUCT TOPOLOGY ON PRODUCT SPACE

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ABSTRACT: Function space is one of the fundamental areas of research in functional analysis. We need to explore some potential topologies in the family of functions F from some arbitrary set X into another set Y with a view to investigate the properties of the various topologies generated on the family of functions F by subsets of X, such as compactness, separability and completeness. From the result of the research, we established the nature or characteristics of the following function space topologies, namely: Product topology $\mathfrak{J}_M$, Point-open-topology $\mathfrak{J}_0$, Topology of point-wise-convergence $\mathfrak{J}_w$, Compact-open topology $\mathfrak{J}_C$, Topology of uniform convergence $\mathfrak{J}_U$, and the Seminorm topology $\mathfrak{J}_P$. Comparing the function space topologies, we established that $\mathfrak{J}_M < \mathfrak{J}_C < \mathfrak{J}_U$. This shows that $\mathfrak{J}_U$ is the strongest. But the three topologies coincide when X is finite. The product topology $\mathfrak{J}_M$ is equally, the Seminorm topology $\mathfrak{J}_P$ of point-wise –convergence on F. If the function space has the topology with the base of the form $\cap {\pi_i}^{-1}(0, \varepsilon)$, we call this topology, the Seminorm topology of uniform convergence on F. Uses were made of the definition of the defining subbase and base of a topology. The concept of compactness of a set,

and the composite mapping were used to establish results. We have also made use of the definition and properties of seminorm to establish results. It was also established that the function space topologies are Hausdorff as each one separates points of X. Uses were made of the Separation Axioms. Not alone, it was established that the function space topologies are locally convex if *F*+ contains zero function. This work contributes to knowledge by having established that "every function space topology is a product topology".

Keywords: *Function Space, Topology, Seminorm, Separability, Compactness*

1.1 INTRODUCTION

Let X and Y be arbitrary sets, and let $\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Y})$ denote the collection of functions from X into Y. Any sub – collection of $\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Y})$ with some topology is called a function space. The function space $\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Y})$ can be **identified with a product set as follows**:

Let Y_a denote a copy of Y indexed by $a \in X$, and let F denote the product of sets Y_a

$$\boldsymbol{i.e\,F} = \pi\{\boldsymbol{Y_a}: \boldsymbol{a \epsilon X}\}$$ **(See Oraekie,2014)**

Recall that F consists of points $p = \{a_i : a_i \epsilon X\}$, which assign to each $a_i \epsilon X$ the element $a_i \epsilon Y_i = Y$.

i.e. F consists of all functions from X into Y, and so $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{X},\boldsymbol{Y})$.

Now for each element $a_i \epsilon X$, the mapping $\boldsymbol{e_i}$ from the function set $\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Y})$ into Y defined by $\boldsymbol{e_i(x)} = \boldsymbol{f(i)}$

is called the evaluation mapping at i.

Here f is any function in $\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Y})$.

$\boldsymbol{i.e.f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$

Under our identification of $\boldsymbol{F}(\boldsymbol{X},\boldsymbol{Y})$ with $\boldsymbol{F}$, the evaluation mapping $\boldsymbol{e_i}$ is precisely the **projection mapping** $\boldsymbol{\pi}$ from F into the coordinate space $\boldsymbol{Y_i} = \boldsymbol{Y}$.

Consider example.

***Example* 1.1**

Let $F\ (I,R)$ be the collection of all real – valued functions defined on $I = [O,I]$,

$and\ let\ f, g, h \in \boldsymbol{F(I, R)}\ be\ the\ functions.$

$F(x) = x^2, \quad g(x) = 2x + 1, \quad h(x) = \sin \pi x$

$Consider\ the\ \boldsymbol{evaluation\ function}\ \boldsymbol{e_i : F(I, R) \to R}, \quad at, say, \quad \boldsymbol{i} = \frac{1}{2}$

$Then, \boldsymbol{e_i(f) = f(i) = f\left(\frac{1}{2}\right)} = \frac{1}{4}, \quad and\ \boldsymbol{i} = 0, \quad f(0) = 0$

$\boldsymbol{e_i(g) = g(i) = g\left(\frac{1}{2}\right) = 2, and\ i = 0, g(0) = 1}$

$\boldsymbol{e_i(h) = h(i) = h\left(\frac{1}{2}\right) = 1\ and\ i = 0, h(0) = 1}$

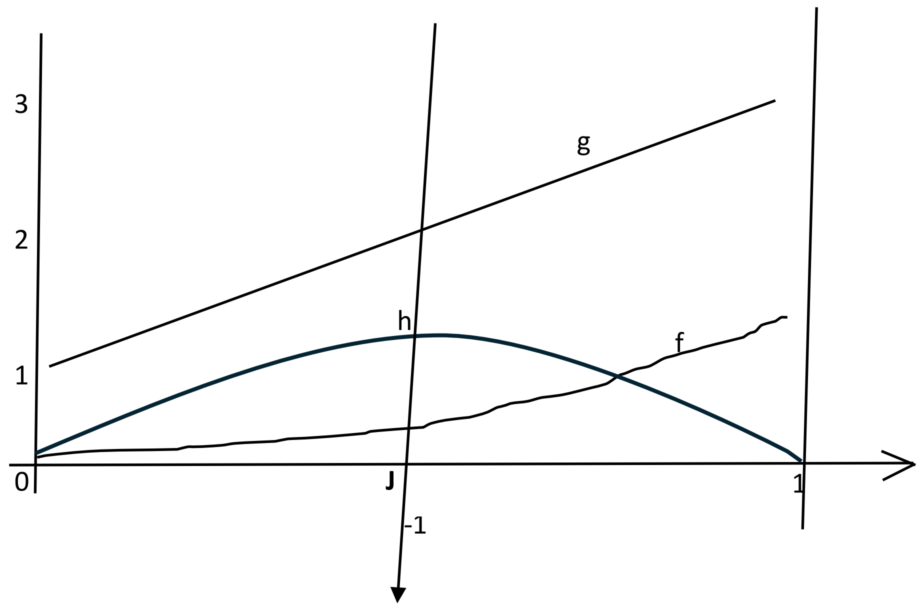

Fig 1.1

R_J

$The\ graphically\ ploting\ gives,$
$e_i(f), e_i(g)\ and\ e_i(h)\ are\ points\ where\ the\ graphs\ of\ f, g\ and\ h\ intersect\ the\ vertical\ line\ R_i\ through\ x = i$
$However, no\ work\ has\ been\ reported\ on\ the\ topological\ properties\ of\ function\ spaces\ such\ as\ compactness\ and\ completeness\ or\ characteristics\ and\ relationship\ with\ the\ family\ as\ a\ function\ space. Hence\ the\ research.$

2. *Preliminaries*

2.1. *Product Space*

$Let\ (X_i,: \ i \in I\ )\ be\ any\ class\ of\ sets\ and\ let\ X\ denote\ the\ Cartesian\ product$

$of\ these\ sets$

$i.\boldsymbol{e}\ \boldsymbol{X} = \boldsymbol{\pi_i X_i, i\epsilon I}.$

$Note\ that\ X\ consists\ all\ points\ p = \{a_i : i\epsilon I\}, where\ a_i \epsilon X_i$

$Recall\ that\ for\ each\ i_0\epsilon I, we\ defined\ the\ projection\ \boldsymbol{\pi_{i_0}}\ from\ the\ product\ set\ X\ to\ the$

$coordinate\ space\ X_{i_0}$

$i.e. \boldsymbol{\pi_{i_0}} : X \rightarrow X_{i_0}\ by\ \boldsymbol{\pi_{i_0}}(\{a_i : i\epsilon I\}) = a_{i_0}$

$These\ projections\ are\ used\ to\ define\ the\ product\ topology\ (see\ Lipschutz\ .S.(1965)$

$2.1.1$. ***Product Topological space or Product Space***

$Let\ (X_i, \mathfrak{I}_i)\ be\ a\ family\ of\ topological\ spaces\ and\ let\ X\ be\ the\ product\ of\ the\ sets\ X_i.$
$ie\ X = \pi_i X_i, i\epsilon I$
$the\ coarsest\ topology\ \boldsymbol{\mathfrak{I}_M}\ an\ X\ with\ respect\ to\ which\ all\ the\ projections$
$\pi_i : X \rightarrow X_i$
$are\ contiuous\ is\ called\ the\ (Tychonoff)$ ***product topology***.
$The\ product\ set\ X\ with\ the\ product\ topology\ \boldsymbol{\mathfrak{I}_M}\ .i.e.(X, \boldsymbol{\mathfrak{I}_M}), is\ called\ the\ product$
$topological\ space\ or\ simply, product\ space.$

Example $2.1.1$

$Consider\ the\ Cartesian\ plane\ \boldsymbol{R^2} = \boldsymbol{R\ x\ R}.$
Recall that the inverse $\boldsymbol{\pi_1}^{-1}(\boldsymbol{a}, \boldsymbol{b})$ ***and*** $\boldsymbol{\pi_2}^{-1}(\boldsymbol{a}, \boldsymbol{b})$ $are\ infinite\ open\ strips$
$which\ form\ a\ subbase\ for\ the\ usual\ topology\ on\ R^2$
$\pi_i\ R^2 = \pi_i\ R_i \rightarrow R$
$thus\ the\ usual\ topology\ on\ R^2\ is\ the\ topology\ generated\ by\ the\ projection\ from\ R^2\ into$

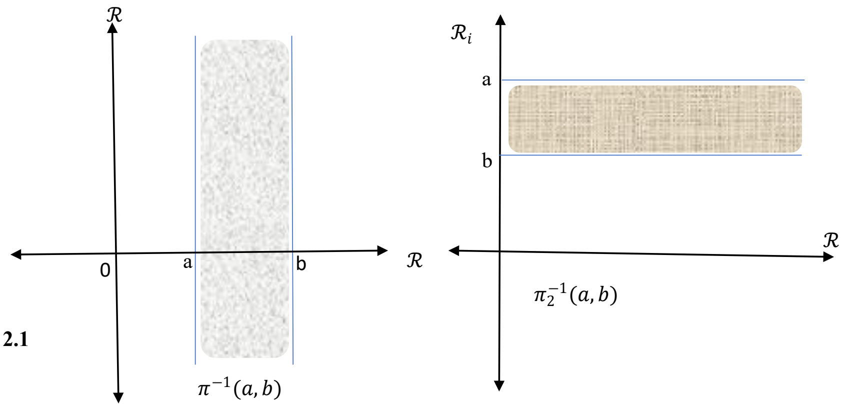

Fig 2.1

Fig. 2.1

Proposition 2.1.1

Product space of Hansdorff topological spaces is Hansdorff.

Proof:

Let $(X_i : i \epsilon I)$ *be a collection of Hansdorff spaces and let* X *be the product space.*

$i.e. X = \pi_i X_i, i \epsilon I,$

We need to show that X *is also a Hausdorff space.*

Let $P = (a_i : i \epsilon I)$ *and* $q = (b_i : i \epsilon I)$*. be distinct point in* X.

Then, P *and* q *must differ in at least one coordinate space, say* $\boldsymbol{X_{i_0}}$

i.e. $\boldsymbol{a_{i_0} \neq b_{i_0}}$

by hypothesis X_{i_0} *is Hansdorff; hence there exist disjoint open sets* **G** *and* **H** *of* X_{i_0}

such that $a_{i_0} \in G$ *and* $b_{i_0} \in H$.

by definition of the product space, the projection $\boldsymbol{\pi_{i_0} : X \to X_{i_0}}$ *is continuous.*

Accordingly

$\boldsymbol{\pi_{i_0}}^{-1}[G]$ *and* $\boldsymbol{\pi_{i_0}}^{-1}[H]$ *are open disjoint Sublet of* X *containing* P *and* q *respectively.*

Thus, X *is a Hansdorff space*

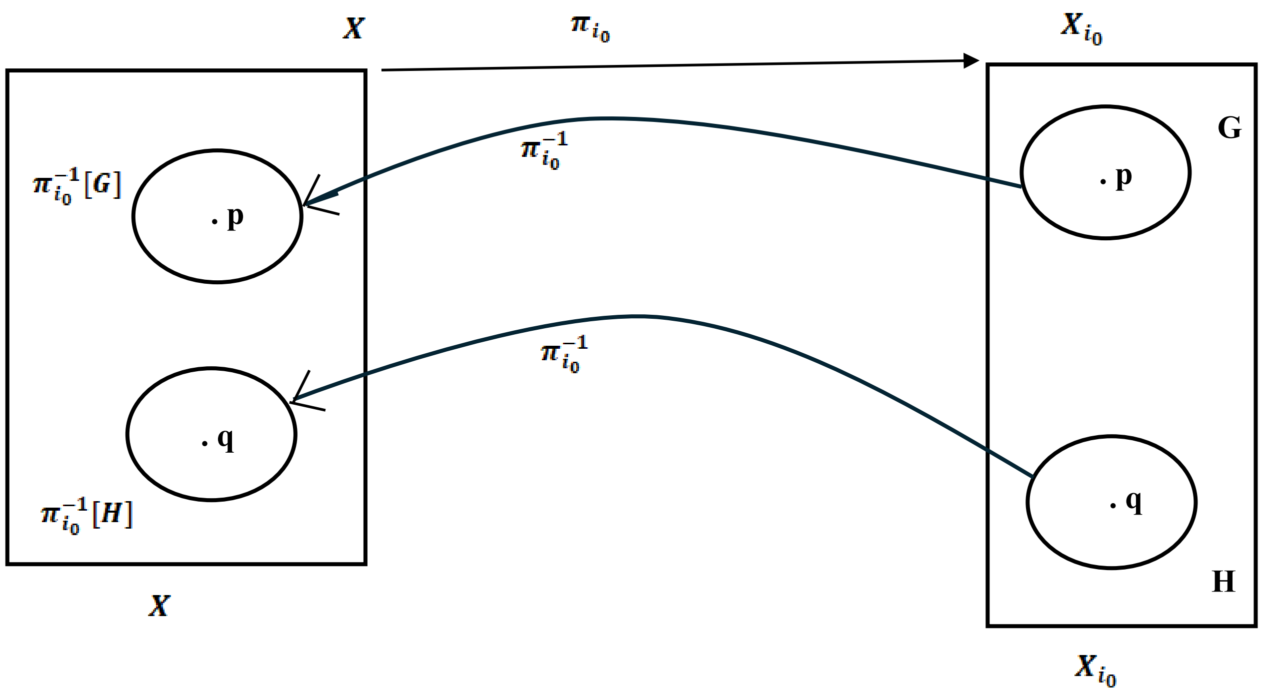

Fig 2.2

2.2 . Topologies Generated by Functions

Let $\{(Y_j, \mathcal{J}_j)\}$ be any collection of topological spaces and for each Y_j, let there be given a function $f_j : X \rightarrow Y_j$ denoted by some arbitrary non − empty set X.

We wish to investigate those topologies an X with respect to which all the functions f_i are continuous.

We are aware that each f_j is continuous relative to some topology on X if and only if the inverse image of each open subset of Y_j is an open subset of $X = F(X, Y_j)$

Now, let us consider the following class of subsets of X:

$$\mathcal{A} = \bigcup_{j \in \Delta} \{\boldsymbol{f_j}^{-1}[\boldsymbol{G_j}] : \boldsymbol{G_j} \subset \mathcal{J}_j\}$$

That is $\mathcal{A}$ consist of the inverse image of each open subset of every space Y_j, $\forall\ j$.

The topology $\mathcal{J}$ on X generated by $\mathcal{A}$ is called the topology generated by the family of functions $\boldsymbol{f_j}$

Properties of the $\mathcal{J}$.

Here are the main properties of the $\mathcal{J}$ as stated :

(i). All the functions f_j are continuous relative to $\mathcal{J}$.

(ii). $\mathcal{J}$ is the intersection of all the topologies on X with respect to which the functions f_j are continuous.

(iii). $\mathcal{J}$ is the smallest, i.e. coarsest topology on X with respect to which the functions f_j are continuous.

(iv). $\mathcal{A}$ is a subbase for the toplogy $\mathcal{J}$.

Thus, we called $\mathcal{A}$ the defining subbase for the topology generated by the functions f_j (Schuptz . S. (1965).

Proof for (i)

For any function $f_j : (X, \mathcal{J}) \rightarrow (Y_j, \mathcal{J}_j)$,

If $H \in J_j$, then $\boldsymbol{f_j}^{-1}[\boldsymbol{H}] \in \mathcal{A} \subset \mathcal{J}$

Hence all f_j are continuous with respect to $\mathcal{J}$

$\boldsymbol{For}$ (ii)

(a) $We\ first\ show\ that\ if\ the\ family\ \{\mathfrak{I}_j\}\ of\ topologies\ on\ a\ set\ X\ is\ given, and$ $if\ a\ function\ f: X\ \rightarrow\ Y, is\ continuous\ with\ respect\ to\ each\ \mathfrak{I}_j, then\ f\ is$ $continuous\ with\ respect\ to\ the\ intersection\ topology$

$$\mathfrak{I} = \bigcap_{i \in \Delta} \mathfrak{I}_j$$

$\boldsymbol{Proof}$

$Suppose\ f: X\ \rightarrow\ Y\ is\ continuous, and\ let\ E\ be\ a\ closed\ subset\ of\ Y,$

$then\ E^C\ is\ open, and\ so\ f^{-1}[E^C\ ]\ is\ open\ in\ X. But$

$f^{-1}[E^C\ ]\ = (f^{-1}[E])^C$

$Therefore, \quad f^{-C}[E]\ is\ closed.$

$Conversely, assume\ E\ is\ \ closed\ in\ Y. then\ it\ implies\ that$

$f^{-1}[E]\ is\ closed\ in\ X.$

$Let\ G\ be\ an\ open\ subset\ of\ Y. then\ G^c\ is\ closed\ in\ Y, and\ so$

$f^{-1}\ [G^C] = (f^{-1}[G])^C\ is\ close\ in\ X.$

$Accordngly\ , f^{-1}[G]\ is\ open\ and\ therefore, f\ is\ continuous.$

(b) $Now\ let\ \mathfrak{I}^*\ be\ the\ intersection\ of\ all\ topologies\ on\ X\ with\ respect\ to\ which\ the$

$functions\ f_j\ are\ continuous.$

$We\ need\ to\ show\ that\ \mathfrak{I} = \mathfrak{I}^*.$

$By\ the\ argument\ above\ (a), all\ the\ functions\ f_i\ are\ also\ continuos\ with\ respect\ to\ \mathfrak{I}^*;$ $hence\ \mathcal{A} \subset \mathfrak{I}^*\ \ and, since\ \mathfrak{I}\ is\ the\ topology\ generated\ by\ \mathcal{A}, \quad \mathfrak{I} \subset \mathfrak{I}^*\ .$

$On\ the\ other\ hand, \mathfrak{I}\ is\ one\ of\ the\ topologies\ with\ respect\ to\ which\ the\ f_j\ \ are$ $continuous;\ hence\ \mathfrak{I}^* \subset \mathfrak{I}.$

$\Rightarrow \mathfrak{I} = \mathfrak{I}^*.$

$\boldsymbol{For}$ (iii)

$\boldsymbol{The\ proof\ follows\ from}$ (ii).

(iv). $The\ proof\ follows\ from\ the\ fact\ that\ any\ class\ of\ sets\ is\ a\ sub\ base\ of\ the$ $topology\ it\ generates.$

2.3. COVERS

Let $\mathcal{A} = \{G_j\}, j \in \Delta$ be a class of subsets of X such that

$$A \subset \bigcup_{j=1} G_j \text{ , for some } \mathcal{A} \subset X.$$

Then $\mathcal{A}$ is called a **cover** of A, and an **open cover** if each G_j is open.

If **a finite subclass** of $\mathcal{A}$ is also a cover of A.

i.e if there exists

$G_{j_1}, G_{j_2}, G_{j_3}, \ldots, G_{j_n} \in A$, such that

$$A \subset G_{j_1} \bigcup G_{j_2} \bigcup G_{j_3} \bigcup \ldots \bigcup G_{j_n}$$

Then $\mathcal{A}$ is said to be **reducible to a finite cover**, or contains a finite subcover.

2.3.1. HEINE – BOREL Theorem

One of the most important properties of **a closed and bounded interval** is given in the Heine – Borel theorem.

Here, a class of sets $\mathcal{B} = \{B_j\}$, is said to cover a set B if B is contained in the union of the members of $\mathcal{B}$,

i.e . $B \subset \bigcup_{j \in \Delta} B_j$

The Theorem (HEINE BOREL THEOREM) 2.3

Let $B = [a, b]$ be a closed and bounded interval, and let $\mathcal{G} = \{G_j : j \in \Delta\}$ be a class of open sets (open intervals) which covers B,

i.e. $B \subset \bigcup_{j \in \Delta} G_j$

Then, $\mathcal{G}$ contains a finite subclass, say $\{G_{j_1}, G_{j_2}, G_{j_3}, \ldots, G_{j_n}\}$ which also covers B,

i.e. $B \subset G_{i_1} \bigcup G_{i_2} \bigcup G_{i_3} \bigcup \ldots \bigcup G_{i_n}$

Proof

(**see Schutz** (1965) **page** 58

Example 2.3

Let $B = [a, b]$ be a closed and bounded interval and let $\{G_j : j \in \Delta\}$ be a class of open sets such that

$$B \subset \bigcup_{j \in \Delta} G_i$$

Then one can select a finite number of the open sets, say

$$\boldsymbol{G_{j_1}, G_{j_2}, G_{j_3}, \ldots, G_{j_n}} \quad \text{so that } \boldsymbol{B \subset G_{j_1} \bigcup G_{j_2} \bigcup G_{j_3} \bigcup \ldots \bigcup G_{j_n}}$$

For this and on this above terminology, the Heine – Borel theorem can be restated as follows:

2.3.1. *HEINE – BORAL THEOREM*:

Every open cover of a closed and bounded interval $B = [a, b]$ *is reducible to a finite cover.*

2.3.2. *COMPACT SET*

A subset B of a topological space X is said to be compact if every open cover of B is reducible to a finite cover.

***Example* 2.3.1**

Let B be any finite subset of a topological space X, say $B = \{b_1, b_2, b_3, \ldots, b_n\}$.

Assuming that $\mathcal{G} = \{G_j\}$ *is an open cover of B,*

Then each points in B belongs to one of the members of G, say

$b_1 \in G_{j_1}, b_2 \in G_{j_2}, b_3 \in G_{j_3}, \ldots, b_n \in G_{j_n}$,

$$\boldsymbol{B \subset G_{j_1} \bigcup G_{j_2} \bigcup G_{j_3} \bigcup \ldots \bigcup G_{j_n}}$$

***Example* 2**

By the Heine – Borel Theorem, every closed and bounded interval $[a, b]$ *on the real line R is compact.*

2.4. *LOCALLY COMPACT SPACE*

A topological space X is said to be locally compact if and only if every point in X has a compact neighborhood.

***Example* 2.4**

Consider the real line R with the usual topology. Observe that each point $p \epsilon R$ *is interior to a closed interval.*

i.e. $[p - \varepsilon, p + \varepsilon]$, *and that the closed interval is compact by the Heine – Bovel Theorem.*

Hence R is locally compact space.

On the other hand, R is not compact space; for instance, the class

$\mathcal{G} = \{(\ldots,(-3,-1),-2,0),(-1,1),(0,2),(1,3),\ldots\}$ *is an open cover of* R, *but contains no finite sub cover.*

Thus we see, by the above example, that a locally compact space need not be compact.

3. *MAIN WORK*

3,1. *PRODUCT SET*

Let X *and* Y *be two sets and let* $\boldsymbol{f}:\boldsymbol{X} \to \boldsymbol{Y}$ *be a function from the set* X *into the set* Y.

let $\mathcal{F}$ *be a family of functions from a set* X *into the set* Y. *we denote* $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(\boldsymbol{X},\boldsymbol{Y})$

and $\boldsymbol{f} = \boldsymbol{f_a} = \boldsymbol{f}(\boldsymbol{a})$ ***for every*** $\boldsymbol{a \epsilon X}$.

Then, put $\boldsymbol{\pi Y_a} = \mathcal{F}, \forall\, \boldsymbol{a \epsilon X}$.

Here $\mathcal{F}$ *becomes a product of copies of* Y.

For instance, if $\boldsymbol{X} = \{\boldsymbol{a_1},\boldsymbol{a_2}\}$.

Then, $\mathcal{F} = Y_{a_1} x Y_{a_2} = \pi Y_{a_i}, \quad \forall a_i \epsilon X, i = 1,2.$ *(a product space)*

Example 3.1

Let $X = \{a_1, a_2, \boldsymbol{a_3}, \ldots, a_n\}$. *Then let* $f: X \to Y$ *such that*

$\mathcal{F} = \{f(a_1), f(a_2), \ldots, f(a_n)\}$

$\Rightarrow \mathcal{F} = \{f(a_i)\} \in \pi_i Y_{a_i}, \quad i = 1,2,\ldots,n$

From the above examples, we realized that, if $f: X \to Y$, *then*

$\boldsymbol{f} \equiv \boldsymbol{f}(\boldsymbol{a}) \in \boldsymbol{\pi Y_a}, \forall\, \boldsymbol{a \epsilon X}, \quad Ya = Y$

Thus, we conclude that $\mathcal{F}$ *is a product space.* $\boldsymbol{\pi Y_a}$ *of copies of* Y *for every* $a \in X$.

3.1.1. *Point − open Topology* $\boldsymbol{\mathcal{J}_0}$ *on the product space* $\boldsymbol{\pi_a Y_a}$ $\forall\ \boldsymbol{a} \in \boldsymbol{X}$

let X *be a set, and* $(Y, \mathcal{J})$ *be a topological space with topology* $\mathcal{J}$.

Let the V *be any open set in* Y, *i.e* $V \in \mathcal{J}$.

Put

$V_a = \{f(a) \in \mathcal{F}$ *such that* $f(a) \in V\}$

Or

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\pi_a}^{-1}[\boldsymbol{V}] = \{\boldsymbol{f} \in \boldsymbol{\mathcal{F}}, \boldsymbol{\pi_a}(\boldsymbol{f}) = \boldsymbol{f}(\boldsymbol{a}) \in \boldsymbol{V}\} \quad \ldots\ldots\ldots(\mathbf{1.1})$$

***Illustration* 3.1**

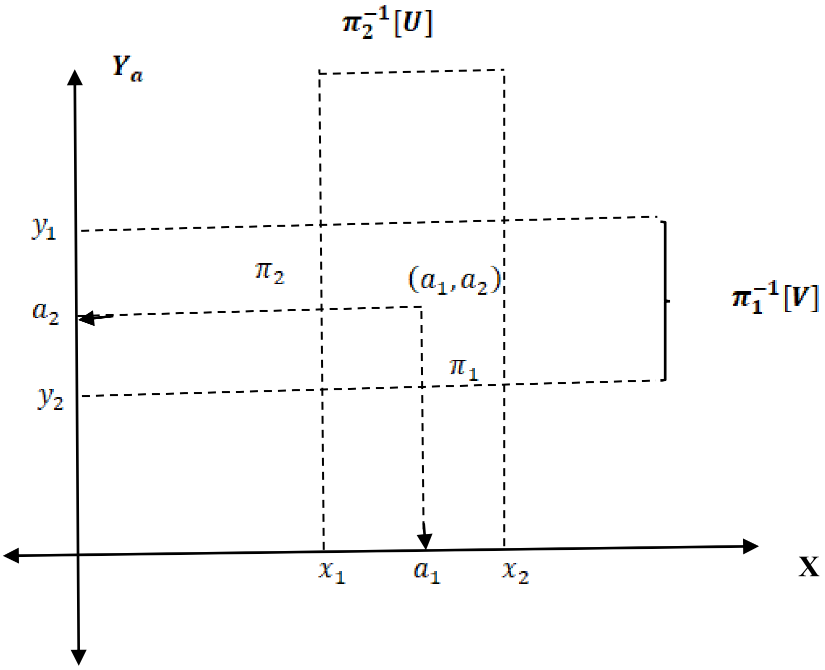

Fig 3

$\boldsymbol{\pi_{a_1}(f) \;=\; f(a_2) \in V = (y_1, y_2), \text{an open interval in } Y}$

$\boldsymbol{\pi_{a_2}(f) \;=\; f(a_1) \in U = (x_1, x_2), \text{an open interval in } X}$

$If\ f \in \mathcal{F}, then\ for\ any\ point\ a \in X, f(a) \;=\; y \in V\ for\ same\ open\ set\ V\ in\ Y.$

$\Rightarrow f \in V_a \quad or \quad f \in \pi_a{}^{-1}[V]$

And

$$\bigcup_{a \in X:\, V \in \mathcal{J}} V_a = \boldsymbol{\mathcal{F}\,; V \in \mathcal{J}}.$$

$Thus, equation$ (1.1)$constitutes\ a\ subbase\ \boldsymbol{\mathcal{A}}\ for\ a\ topology\ on\ \boldsymbol{\mathcal{F}} as\ a\ varies\ in\ X$ $and\ V\ varies\ in\ \mathcal{J}. Then, let\ the\ base\ \beta\ for\ the\ topology\ \mathcal{J}\ on\ \mathcal{F}\ be\ denoted\ by$

$\boldsymbol{\beta = \{finite\ intersections\ of\ elements\}\ of\ \mathcal{A}}$

$If\ V_a\ is\ an\ open\ subset\ of\ Y_a, then$

$\boldsymbol{\pi_a{}^{-1}[V_a]} \subset \boldsymbol{\mathcal{F}}, a\ member\ of\ \mathcal{A}.$

$Here, \pi_a\ is\ the\ \boldsymbol{ath}\ projection\ map\ from$

$\mathcal{F} = \pi_a Y_a \quad onto\ \ the\ coordinate\ Y_a, \forall\ \boldsymbol{a \epsilon X}$

$as\ it\ is\ illustrated\ in\ fig$ 1.1 $above,$

$showing\ that\ \mathcal{J}_0\ is\ also\ a\ product\ topology$

$on\ \pi_a Y_a, \qquad \forall\, \boldsymbol{a} \in \boldsymbol{X}$

$Thus, the\ Point - open\ Topology\ \mathcal{J}_0\ on\ F\ is\ the\ product\ topology\ \mathcal{J}_m\ an$

$\mathcal{F} = \pi_a Y_a, \qquad \forall\, \boldsymbol{a} \in \boldsymbol{X}$

$for\ which\ all\ the\ projections\ \pi_a\ , a\epsilon X\ are\ continuous.$

$\boldsymbol{This\ is\ because\ the\ sets\ in\ the\ subbase\ \mathcal{A}\ are\ inverse\ images\ of\ open\ sets\ in\ the\ coordinate\ spaces\ by\ the\ projection\ maps\ \pi_a}$

$Now\ suppose\ that\ B\ is\ a\ member\ of\ the\ base\ for\ the\ product\ in\ an\ infinite\ product\ space\ \mathcal{F}, where\ \mathcal{F} = \pi_x R_x\ such\ that\ x \in X(of\ copies\ of\ R), then\ for\ all\ x \in X\backslash U.$
$;\ where\ U\ is\ a\ finite\ subset\ of\ X)\ .$
$we\ have\ this\ to\ say,$

$\pi_a(B) = R$

$when\ a\ is\ not\ an\ element\ of\ U\ and\ when\ b \in U, then$

$\boldsymbol{\pi_b(B) = (p - \varepsilon_b\ , p + \varepsilon_b), for\ som\ b \in X\ , p \in R\ \ with\ \ \varepsilon_b > 0}.$

$If\ we\ are\ given\ any\ point\ a \in X, and\ f_1, f_2, f_3, \ldots .. \in F\ converge\ in\ (\mathcal{F}, \mathcal{J})$

$i.e.\ f_1(a), f_2(a), \ldots, f_n(a) \ldots \to g(a)\ in\ the\ coordinate\ space\ \ Y_a = Y$

$\Rightarrow convergence\ under\ \mathcal{J}\ is\ the\ usual\ convergence\ in\ Y_a\ by\ the\ projection\ map\ \pi_a\ for\ a\ fixed\ a \in X.$
$Not\ alone, supposed\ also\ that\ if\ \ b \in X$
$f_1(b), f_2(b), f_3(b), \ldots \ldots . \quad \to \quad g(b) \in Y_b = Y$

$\Rightarrow f_1, f_2, f_3, \ldots .. \to g\ \ in\ \ (\mathcal{F}, \mathcal{J})$

$We\ observed\ that\ this\ convergence\ is\ point\ by\ point. And\ so\ we\ call\ \boldsymbol{\mathcal{J}_0}$ the topology of pointwise – conergence $\mathcal{J}_w$
$Thus, \mathcal{J}_w\ is\ the\ topology\ of\ pointwise\ convergence.$

$Now\ suppose\ that\ for\ any\ x\ in\ X, it\ is\ true\ that\ \{f_n(x)\}\ coverges\ to\ f(x)\ in\ R_x\ ,$

$where\ n = 1,2, \ldots \quad Then\ it\ wll\ be\ true\ that\ \{f_n\}\ converges\ to\ f\ in\ (\mathcal{F}, \boldsymbol{\mathcal{J}_0}\ ).$

$Conversely, suppose\ that\ \{f_n\}\ converges\ to\ f\ in\ (\mathcal{F}, \boldsymbol{\mathcal{J}_0}\ ).$

$Let\ y\ be\ any\ point\ of\ X\ and\ u\ be\ any\ basc\ open\ set\ \ in\ R_y\ containing\ f(y). Then$

$$T = \pi_{x \in X} u_x$$

(*where* u_x

$= u$ *and* u_x *is open for all* x) *is a basic open set in* $(\mathcal{F}, \mathfrak{J}_0)$ *containing* f

and so cotains $\{f_n\}, n \geq N_0$ *say.*

Now, let $V = \pi_{x\epsilon X} V_x$, where $V_x = (p_x - \varepsilon, p_x + \varepsilon)$, $p_x \in R_x$ for all $x \in X$.

Now V_x *is not open in* $\mathfrak{J}_0$, *but is open in the next topology we shall consder.*

From our terminology we have that,

the point open topology $\mathfrak{J}_0$ on $\mathcal{F}$

$\equiv$ **the product topology $\mathfrak{J}_m$ on $\mathcal{F}$**

$\equiv$ **the topology of pointwise – conergence $\mathcal{J}_w$ on $\mathcal{F}$**

$\equiv$ **the weak topology ω on $\mathcal{F}$.**

i.e.

$$\mathfrak{J}_0 = \mathfrak{J}_m = \mathcal{J}_w = \omega.$$

***Theorem* 3.1. (*Properties of Point Open – Topology*).**

Let X *be on set, and* $(X, \mathcal{J})$ *be a topological space with the topology* $\mathcal{J} \subset X$.

Let $\mathcal{F}$ *the family functions of* $f: X \rightarrow Y$, *from* X *into* Y.

The point open topology on $\mathcal{F}$ *is Hausdorff.*

Proof

Let X *be a set,* $(X, \mathcal{J})$ *be a topological space. Let* $f, g \in \mathcal{F}$ *be functions such that*

$f \neq g$ *and there exists a point* $a \in X$ *such that*

$f(a) \neq g(a)$.

This implies that there must be two disjoint open set H *and* G *in* Y *such that*

$f(a) \in H, g(a) \in G$, *such that*

$f \in Ha$, $g \in Ga$ *and* $H_a \cap Ga = \emptyset$

Where, $H_a = \pi_a{}^{-1}(Ha)$ *and* $G_a = \pi_a{}^{-1}(G_a)$ *and each function in* $\mathcal{F}$ *is*

many – to – one.

3.2. *The Compact Open Topology on the family of functions* $\mathcal{F} = \pi_a Y_a$; $a \in X$

let X *and* Y *be arbitrary sets such that* $A \subset X$ *and* $B \subset Y$ *respectively.*

Let us denote

$$\mathcal{F}(A, B) = \{f \epsilon \mathcal{F} : f(A) \subset B\}$$

For all family of functions from X into Y such that each function maps A into B.
Consider an example

Examples 3.2

Let $\mathcal{A}$ be the defining subbase for the pont − open topology on $\mathcal{F}$.

Since each member of $\mathcal{A}$ is of the form.

$\{f\epsilon\mathcal{F}: f(a) \subset V\}$ (3.2)

For some fixed $a\epsilon X$ and a fixed open set V in Y.
For this reason, we could denote the set (3.2) by

$\mathcal{F}(a,V) = \{a\epsilon X: f(a) \subset V\}$.

Thus, we could define the subbase $\mathcal{A}$ by

$\mathcal{A} = \{\mathcal{F}(a,V): a\epsilon X \text{ and } V \subset Y\}$ (3.3)

Suppose that X and Y are topological spaces and let $\boldsymbol{S}$ be the class of compact subsets of X, and let $\boldsymbol{G}$ be the class open subsets of Y.

The topology on $\mathcal{F} = \mathcal{F}(X,Y) = \pi_a Y_a: a\epsilon X$, generated by

$\mathcal{A} = \{\mathcal{F}(E,V): E \in \boldsymbol{S}, V \subset G\}$

as a subbase is called the **compact open topology** $\mathcal{J}_c$ **on**

$\mathcal{F}(X,Y) = \mathcal{F} = \pi_a Y_a \;;\; a\epsilon X$

But the singleton subsets of X are compact, therefore, $\mathcal{A}$ contains all members of the defining subbase for the point − open topology $\mathcal{J}_0$ on

$\mathcal{F}(X,Y) = \mathcal{F} = \pi_a Y_a \;;\; a\epsilon X$

From the above, we conclude that if X and Y are topological spaces, then the point − open topology $\mathcal{J}_0$ on $\mathcal{F}(X,Y) = \mathcal{F} = \pi_a Y_a \;;\; a\epsilon X$ is coarser than the compact − open topology $\mathcal{J}c$ on F.

3.3. The Topology of Uniform Convergence $\mathcal{J}_U$ on

$\mathcal{F}(X,Y) = \mathcal{F} = \pi_a Y_a \;;\; a\epsilon X$

Let f be a function from X into Y where $(X,\mathcal{J})$ be a topological space and $Y = R$ such that $f: (X,\mathcal{J}) \to R$.
Let $\mathcal{F} = \mathcal{F}(X,R) = \pi_a R_a : a\epsilon X$ ($R_a = R$). Where R is a real number

$As\ in\ the\ case\ of\ the\ point - open - topology\ \mathcal{J}_0\ on\ the\ product$
$\mathcal{F} = \pi_a R_a : a\epsilon X.\ \ Where\ R_a = R\ for\ all\ a\epsilon X$
$We\ wish\ to\ construct\ \boldsymbol{a\ topology\ of\ uniform\ convergence}\ \mathcal{J}_u\ on\ \mathcal{F}\ such\ that\ the$
$sequence\ of\ functions\ \{f_n\}\ converges\ to\ f\ in\ this\ topology\ if\ and\ only\ if\ \pi_a(f_n)$
$converges\ to\ \pi_a(f)\ in\ R_a\ for\ all\ a\epsilon X$
$i.e.\pi_a : \mathcal{F} \to R_a$
$is\ the\ \boldsymbol{ath}\ projection\ from\ the\ product\ space\ \mathcal{F}\ onto\ R_a \equiv R$
$the\ ath\ coordinate\ space.$
$Since\ R_a$
$= R\ is\ endowed\ with\ the\ usual\ topology, then\ a\ base\ \boldsymbol{\mathcal{B}_u}\ of\ \mathcal{J}_u\ consists\ of$
$sets\ of\ the\ form$

$$\bigcap_{a\epsilon X} \boldsymbol{\pi_a}^{-1}(\boldsymbol{r_a} - \boldsymbol{\varepsilon_a}, \boldsymbol{r_a} + \boldsymbol{\varepsilon_a}),$$

$Where\ \boldsymbol{r\epsilon R}, \boldsymbol{\varepsilon_a} > \boldsymbol{o}\ and\ finite.$
$Here, the\ topology\ of\ uniform\ convergence\ \mathcal{J}_u\ on\ \mathcal{F}, every\ projection\ \pi\ of\ any$
$member\ of\ the\ base\ \boldsymbol{\mathcal{B}_u}\ must\ be\ bounded\ in\ R_a = R.$

Proposition 3.3

$Let\ f: X \to R\ be\ a\ function\ from\ X\ into\ R. Where\ X\ is\ a\ topological\ space$
$and\ let\ \mathcal{F} = \mathcal{F}(X, R)\ be\ a\ family\ of\ functions\ from\ X\ into\ R.$
$Then, the\ compact - open\ topology\ \mathcal{J}_C\ on\ \mathcal{F}\ is\ weaker\ than\ the\ topology\ of$
$uniform\ convergence\ \mathcal{J}_U\ on\ \mathcal{F}.$

Proof

$We\ need\ to\ show\ that\ given\ any\ subbasic\ open\ set\ G\ in\ \mathcal{J}_C, then\ G\ is\ open\ in\ \mathcal{J}_u.$
$i.e.G \in \mathcal{J}_C, \Rightarrow G \in \mathcal{J}_u.$
$Now, let\ E\ be\ any\ compact\ subset\ of\ X\ and\ H\ an\ open\ set\ in\ R.$
$Recall\ that\ a\ subbase\ for\ \boldsymbol{compact - open - topology\ on\ \mathcal{F}}\ is\ given\ as$
$\boldsymbol{\mathcal{A}} = \{\boldsymbol{F(\mathcal{S}, V) : E \epsilon \mathcal{S}, V \subset \mathcal{G}}\},$
$Where\ \boldsymbol{\mathcal{S}}\ is\ a\ class\ of\ compact\ subsets\ of\ X\ and\ \boldsymbol{\mathcal{G}}\ the\ class\ of\ open\ subsets\ of\ Y.$
$Thus,$
$\boldsymbol{\mathcal{G} = \{f \epsilon \mathcal{F} : f(E) \subset H\}\ is\ a\ subbasic\ open\ set\ in\ \mathcal{J}_C}.$
$\Rightarrow \{f \epsilon \mathcal{F} : f(E) \subset H\}\ or,$
$\pi^{-1}[H] = (f \epsilon \mathcal{F} : \pi(f) = f(E) \epsilon H)$

$$= \bigcap_{a \epsilon E} \{f \epsilon \mathcal{F} : f(a) \epsilon H_a = H\}$$

$$= \bigcap_{a \epsilon E} f \epsilon \mathcal{F} : \pi_a(f) \epsilon Ha = \pi_a H_a;$$

$for\ every\ a \epsilon X.$ (an open set in $\mathcal{J}_u$)

This completes the proof.

Thus, we established that $\mathcal{J}_C$ is coarser than $\mathcal{J}_u$ (or $\mathcal{J}_u$ contains $\mathcal{J}_C$).

i, e, $\mathcal{J}_C < \mathcal{J}_u$.

3.4. ILLUSTRATION / EXAMPLES

Example 3.4.1

Let $\mathcal{F} = \pi_a R_a, \forall\ a \epsilon R$ be the product of infinite copies of R indexed by R.

Let the open interval $(-\varepsilon, \varepsilon)$ in R, where $\varepsilon > 0$ is a number in $R = R_a$ be $(-\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})\boldsymbol{a}$.

Then the subset $\boldsymbol{G} = \boldsymbol{\pi}_a (-\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})\boldsymbol{a}$ of $\mathcal{F}$ is open in the topology of uniform

convergence in $\mathcal{F}$ but not open in the point − open − topology on $\mathcal{F}$.

We also observed that the three function space topologies namely:

1. **Point − open − topology $\mathcal{J}_0$, or topology of pointwise − convergence $\mathcal{J}_W$**
2. **Compact − open − topology $\mathcal{J}_C$**
3. **Topology of uniform convergence $\mathcal{J}_U$ coincide whenever X is a finite set.**

Example 3.4.2

Let $\mathcal{F}$ be the family of functions f from R into R ($\boldsymbol{f} : \boldsymbol{R} \to \boldsymbol{R}$).

Then there is a sequence $\{f_n\}_{n=1}^{\infty}$ of function in $\mathcal{F}$ which is pointwise convergent to f, but not uniformly convergent to f.

For instance, let $\{\boldsymbol{f}_n\}_{n=1}^{\infty}$ be the following sequence of functions in $\mathcal{F}(R, R) = \mathcal{F}$ defined by

$$f_n(x) = \begin{cases} \mathbf{1} - \dfrac{\mathbf{1}}{\boldsymbol{n}}|\boldsymbol{x}|, \boldsymbol{if}\ |\boldsymbol{x}| < \boldsymbol{n}. \\ \mathbf{0}, \boldsymbol{if}\ |\boldsymbol{x}| \geq \boldsymbol{n} \end{cases}$$

The sequence $\{f_n(x)\}_{n=1}^{\infty}$ *is said to* **converge uniformly** *to a function*

$g: R \rightarrow R.$

if, for every $\varepsilon > 0$, *there exists* $n_0 \epsilon N$ *such that* $n > n_0$ *and* ε *depends on* n_0.

Implies that, $\quad |f_n(x) - g(x)| < \varepsilon$, *for every* $x \epsilon R$

Now the sequence $\{f_n(x)\}_{n=1}^{\infty}$ *converges pointwise to the constant function* $g(x) = 1$.

But $\{f_n\}$ *does not converge uniformly to* g.

For instance, let $\varepsilon = \frac{1}{2}$.

Then for every $n \epsilon N$, *there exists point* $x_0 = n$ *in* R *with* $f(x_0) = 0$

$$i.e. f(x_0) = f(n) = \begin{cases} 1 - \frac{1}{n}|x|, \text{if } |x_0| < n. \\ 0, \text{if } |n| \geq n \end{cases}$$

$$\Rightarrow f(n) = \begin{cases} 1 - \frac{1}{n}|n|, \\ 0, \end{cases} = \begin{cases} 0, \\ 0, \end{cases} = 0$$

So $|f_n(x_0) - g(x_0)| = |0 - 1| = 1 > \varepsilon.$

Example **3.4.3**

Let $f_1, f_2, f_3, f_4, \ldots$ *be the sequence of functions from* $I = [0,1]$ *into* R *defined by* $f_1(x) = x, \quad f_2(x) = x^2, f_3(x) = x^3 \ldots, f_n(x) = x^n$.

Converges pointwise to the function $f : [0,1] \rightarrow R$ *defined by*

$$f(x) = \begin{cases} 0, \text{if } 0 \leq x \leq 1 \\ 1, \quad \text{if } x = 1 \end{cases}$$

We observed that each of the functions f_n *is continuous. But the limit function* f *is not continuous, thus* $\{f_n\}$ *does not converge uniformly to* f.

3.5. *FUNCTION SPACE AS A SEMINORMED LINEAR*

Let $\mathbf{P} = \{p_i(x)\}$ *be a family seminorms on a linear space* X .

$and\ let\ (\boldsymbol{X}, \mathbf{p})\ be\ the\ seminormed\ linear\ space\ determined\ by\ \mathbf{p}\ for\ \boldsymbol{i}\ =\ \mathbf{1}, \mathbf{2}, \ldots$

$Let\ \mathcal{F} =\ \boldsymbol{\pi}_i\, \boldsymbol{R}_i{}^+ : \boldsymbol{i}\epsilon\Delta\ \ be\ the\ product\ of\ copies\ of\ the\ nonnegative$

$real\ numbers\ indexed\ by\ \Delta$

$For\ each\ element\ a\ \epsilon X, consider\ the\ functioning\ \boldsymbol{g} : \boldsymbol{X}\ \rightarrow\ \ \mathcal{F}^+\ defined\ by$

$\boldsymbol{g}\{\boldsymbol{P}_i\,(\boldsymbol{a})\}; \boldsymbol{i}\epsilon\Delta\ a\ point\ in\ \ \mathcal{F}^+$

$The\ function\ \boldsymbol{g}\ may\ not\ be\ one - to - one;\ for\ since\ X\ is\ linear\ , we\ must\ have$

$\boldsymbol{g}(\boldsymbol{a}) = -\boldsymbol{g}(\boldsymbol{a}), \qquad for\ any\ such\ g.$

$Let\ \boldsymbol{\pi}_i : \mathcal{F}^+ \rightarrow\ \ \boldsymbol{R}_i\ be\ the\ projection\ map\ from\ \mathcal{F}^+\ onto\ R_i{}^+$

$We\ observed\ that$

$P_i(a) =\ (\pi_i\ og)(a), a\ \epsilon\ X;$

$since\ (\pi_i\ og)(a)\ =\pi_i\ (g(a)\ =\ \pi_i\ (\{P_i(a)\}\,) =\ P_i(a),\ \ i\ \varepsilon\ \Delta.$

Let

$\boldsymbol{U}_{\boldsymbol{P}_i}\,(\mathbf{0}, \boldsymbol{\varepsilon}) =\ \{\ \boldsymbol{a}\epsilon\boldsymbol{X} :\ \boldsymbol{P}_i(\boldsymbol{a}) < \boldsymbol{\varepsilon}\}\ an\ open\ subset\ of\ X\ ; \forall\ i\epsilon X.$

$Then$

$$\boldsymbol{g}\left\{\boldsymbol{U}_{\boldsymbol{p}_i}\,(\mathbf{0}, \boldsymbol{\varepsilon})\right\}\ =\ \{\boldsymbol{f}\ \in\ \mathcal{F}^+ : \boldsymbol{\pi}_i(\boldsymbol{f}) \in (\mathbf{0}, \boldsymbol{\varepsilon}) \subset \boldsymbol{R}_i{}^+\ =\ \boldsymbol{R}\}$$
$$=\ \boldsymbol{\pi}_i{}^{-1}\,(\mathbf{0}, \boldsymbol{\varepsilon}) \ldots\ldots\ldots\ldots\ldots \qquad (\,\mathbf{1})$$

$Where\ ((\mathbf{0}, \boldsymbol{\varepsilon})\ \subset \boldsymbol{R}_i{}^+\ =\ \boldsymbol{R}^+)\ is\ an\ open\ set\ in\ the\ product\ topology\ of\ \mathcal{F}^+$

$Now, the\ family\ \left\{\boldsymbol{U}_{\boldsymbol{P}_i}\,(\mathbf{0}, \boldsymbol{\varepsilon})\ ; \boldsymbol{i}\epsilon\Delta :\ \ \boldsymbol{\varepsilon} > \mathbf{0}\right\}$ and its translates consttute the defnng

subbase for the topology on X$determined\ by\ the\ family\ \{\boldsymbol{P}_i\}, \boldsymbol{i}\ \epsilon\ \Delta\ \boldsymbol{of\ seminorms}.$

$The\ right\ handside\ of\ (\mathbf{1})\ and\ its\ translates\ constitute\ the\ defining\ subbase\ for$

$the\ product\ topology\ on\ the\ product\ space\ \mathcal{F}^+\ (\boldsymbol{Oraekie}\ \mathbf{2014}).$

$Then, if\ X\ has\ the\ seminorm\ topology, we\ observe\ the\ following\ relationships:$

$(\boldsymbol{a}).\,\boldsymbol{g\ is\ an\ open\ mapping}$

$\boldsymbol{i.\,e.\,g\ maps\ open\ sets\ onto\ open\ sets\ when\ X\ has\ the\ seminorm\ topology}\ \mathfrak{I}_{\boldsymbol{p}}$

$\boldsymbol{and\ \mathcal{F}^+ has\ the\ product\ topology\ \mathcal{J}_m}$.

$\boldsymbol{(b). For\ each\ \pi_i, i\ \epsilon\Delta\ (the\ projection\ map\ from\ the\ product\ space\ \mathcal{F}^+ onto}$

$\boldsymbol{the\ coordinate\ space\ R_i{}^+, = R^+, the\ map\ \pi_i\ o\ g : X \to R_i{}^+ \ is\ continuous,}$

$\boldsymbol{where\ X\ has\ seminorm\ topology\ \mathcal{J}_p}$.

$\boldsymbol{(c). Since\ the\ composition\ map\ g : X \to \mathcal{F}^+ and\ \pi_i : \mathcal{F}^+ \to R_i{}^+}$

$\boldsymbol{is\ continuous\ for\ each\ i\ \epsilon\Delta;\ we\ deduce\ (see\ Liposchutez\ .S. (1965)\ as\ it\ is}$

$\boldsymbol{contained\ in\ Oraekie\ (2014)\ that\ g\ must\ be\ continuous.}$

$From\ (\mathbf{1})\ it\ follows\ that\ g\ is\ continuous\ and\ open.$

$Now, consider\ the\ system$

$$\bigcap_{i\epsilon\Delta}\{U_{P_i}\ (0,\varepsilon));\ \Delta\ a\ definite\ index\ set\}$$

$$= \bigcap_{i\epsilon\Delta}\{ f\ \epsilon\mathcal{F}^+ : \pi_i\ (f)\ \epsilon(0,\varepsilon) \subset R_i{}^+ = R^+\}$$

$$= \bigcap_{i\epsilon\Delta}\pi_i{}^{-1}(0\,,\varepsilon) \qquad (2)$$

$\left(Where\ (0,\varepsilon) \subset R_i{}^+ = R^+\right) is\ an\ open\ set\ in\ the\ product\ topology\ \mathcal{J}_m\ on\ \mathcal{F}^+$.

$We\ observed\ that\ \mathbf{p}_n \to \mathbf{p}\ in\ the\ sense\ of\ the\ product\ topology\ on\ \mathcal{F}^+$ means $\mathbf{p}_n(a) \to \mathbf{p}(a)\ (convergence\ in\ the\ usual\ sense\ of\ scalar).$

$Let\ X\ has\ the\ seminorm\ topology\ and\ let\ \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \ldots\ in\ X\ converge\ to\ \boldsymbol{a}\ in\ X$.

$\boldsymbol{i.e.\{ a_n \}^{\infty}{}_{n=1} \to a}$

$We\ observed\ that$

$$\boldsymbol{p_i(a) = (\pi_i\ og)(a), a\ \epsilon X, for\ every\ i\ \epsilon\ \Delta.}$$

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[dr, "\pi_i o g"'] & \pi_i \mathcal{R}_i^+ = \mathcal{F}^+ \arrow[d, "\pi_i"] \\
& \mathcal{R}_i^+ = \mathcal{R}^+
\end{tikzcd}
$$

Fig. 3.8

$$(\boldsymbol{\pi}_{\boldsymbol{i_1}} og)(a_1) \rightarrow R_{i_1}{}^+$$

$$(\pi_{i_2}\, og)(a_2) \rightarrow R_{i_2}{}^+$$

$$.\,(\pi_{i_3}\, og)(a_3) \rightarrow R_{i_3}{}^+$$

$$\cdots$$

$$(\pi_{i_n}\, og)(a_n) \rightarrow R_{i_n}{}^+$$

$\Rightarrow (\pi_{i_n}\, og)(a_n) \rightarrow (\pi_i\, og)(a)$ *in* $R_i{}^+$ *for each* $i \in \Delta$ *and that*

$$P_i\,(a_n) \rightarrow (\pi_i og)(a) = P_i(a), \textit{for each } i \in \Delta.$$

and on this, we call the topology $\boldsymbol{\mathcal{J}_0}$ *the seminorm topology of pointwise*

Fig. 3.7

convergence of $\boldsymbol{\mathcal{F}}^+$.

Not alone, if $\boldsymbol{\mathcal{F}}^+$ *has the topology with base sets of the form*

$$\bigcap_{i \in \Delta} \boldsymbol{\pi_i}^{-1}[(\mathbf{0}\,,\boldsymbol{\varepsilon})].$$

We call this topology, the ***seminorm topology of uniform convergence*** *on* $\boldsymbol{\mathcal{F}}^+$

REFERENCES

1. Chidume C.E (1989), Functional Analysis; An Introduction, Longman, Nigeria

2. Paul Anaeto Oraekie (2014): Dual Topologies Generated by Polar of Sets, Journal of Applied Sciences, Vol 17 Number 1, Page 11130-11147

3. Obienu, E.U. (1979)" Contributions to The Theory of Linear Topological Spaces over valued field" Ph.D. Thesis University of Ibadan, Nigeria.

4. Oraekie, Paul Anaetodike: (2014), The Topology Determined by the Family of Semi Norms, Journal of Science, Engineering and Technology, Volume 21, Number 2 Pp. 11752 -11765.

5. Larsen, R. (1965) Functional Analysis, Marcel Dekker Inc., New York

6. Prof. P. A Oraekie and N.C. Okafor (2025), The Effects of the Gauge of Convex Balanced and Absorbing Subsets of Linear Topological Space (X, J) on Locally Convex Linear Topological Spaces.

7. Lipchitz. S. (1965), General Topology, Temple University, Tampa (Philadelphia).

8. Waelbroeck. L. (1971), Topological Vector Spaces and Algebra, Springer-Verlag, Berlin